Einstein Equations
from/as Thermodynamics of Spacetime

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Abstract There are two results in the literature that seem closely related: Padmanabhan's interpretation of field equations near a null surface as a thermodynamic identity and Jacobson's derivation of Einstein equations from the Clausius relation of thermodynamics. I compare and contrast these two results in the framework of Gaussian null coordinates near a null surface.

Prologue

A STORY .....

In the court of Akbar Badshah ('Badshah', loosely translated, means 'emperor'), there was a musician called Tansen. He used to enthrall everyone in Akbar's Court with his superb performances. Once, after such a rendition, Akbar started praising him sky-high and said, "There can be no-one else in this world who can sing so well". Tansen disagreed, saying he knows of a hermit who lives in the jungle on the banks of Yamuna river who is far superior and that Tansen himself has learnt music from him for sometime. Akbar, who could not believe this, wanted to listen to this hermit in order to judge for himself. Since the hermit did not want any publicity, it was decided that Tansen will take Akbar near the place where the hermit lived and they should listen to his music without creating any disturbance.

They set out one day and reached the jungle near the river Yamuna, where, at a distance, they saw the hermit's hut. As the sun was setting on Yamuna, with all Nature at peace, the hermit came out his hut, sat on a rock facing the river and started singing. Akbar could immediately see that this was music of a completely different class which Tansen could never produce.

On their way back, Akbar queried, "Tansen, you say he taught you music; clearly, he has held back some techniques from you".

"No", said Tansen. "I know all the technical aspects of music he does."

"But, Tansen, then how do you account for such difference in quality ?"

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"It is simple. He sings for Yamuna while I sing for Badshah".

Source- Paddy's website
My first encounter- 2nd year Undergraduate

1 Introduction

I shall refer to Prof. Padmanabhan as Paddy, as he is popularly known, in this article[1].

Emergent gravity paradigm has been Paddy's main research interest for the past decade [19, 21, 23]. The main motivation for this program is the observation that quantities that look and feel very much like thermodynamic quantities keep popping up in gravity all the time. Hence, it natural to ask "if gravity is the thermodynamic limit of the statistical mechanics of certain microscopic degrees of freedom ('atoms of space') [21]". One of the main results of this research program is the demonstration that Einstein's equations when projected on a null surface can be interpreted as a thermodynamic identity [18, 12, 11, 2, 1, 10]. As Paddy states, "If gravity is thermodynamic in nature, then the gravitational field equations must be expressible in a thermodynamic language [21]." The above results are the concrete realizations of this expectation. There is another result in the literature that smells very similar. Twenty years back, Jacobson showed that enforcing a thermodynamic identity on a local causal horizon is equivalent to enforcing Einstein's equations [8]. This work has also been followed up [3, 5, 7, 15, 4]. The question is often asked: What is the relation between these two approaches? In fact, this question was posed to me during my PhD defence! It is the purpose of this article to answer this question by comparing and contrasting the two results. Let me state at the outset that the two results are not the same result interpreted in two different ways, but arise from two different components of the Einstein equations at the null surface [2, 1]. In particular, the $T\delta S$ terms that appear in the two results are not the same. In Jacobson's case, the change in entropy is as you move along the null geodesics *on* the null surface. In Paddy's case, the change is as you move along the null geodesics *off* the null surface (along the auxiliary null vector).

I shall confine myself to Einstein's theory, and to four dimensions, in this article. For the comparison of the two approaches, I shall use Gaussian null coordinates (GNC) near a null surface [16, 17, 24].

The conventions used in this article are as follows: We use the metric signature $(-,+,+,+)$. The fundamental constants G, $\hbar$ and c have been set to unity. The Latin indices, $a, b, \ldots$, run over all space-time indices, and are hence summed over

[1] This nickname ensured that PhD work was never far from my mind even while enjoying holidays in my home-state of Kerala, because of all the paddy fields around.

four values. Greek indices, $\alpha, \beta, \ldots$, are used when we specialize to indices corresponding to a codimension-1 surface, i.e a $3-$surface, and are summed over three values. Upper case Latin symbols, $A, B, \ldots$, are used for indices corresponding to two-dimensional hypersurfaces, leading to sums going over two values.

2 Comparing the Two Results in GNC

In order to compare Jacobson's and Paddy's approaches, we introduce a coordinate system adapted to a null surface. This coordinate system, named the Gaussian null coordinates (GNC), is quite general like the Gaussian normal coordinates [28] near a non-null surface. Hence, as far as we know, we are not imposing any restrictions on the null surfaces or the spacetimes by restricting to GNC. Further, the quantities that we will be referring to will be physical quantities that do not depend on the choice of coordinates. So, our results derived in the framework of GNC are general results valid around any null surface. The discussion of both Paddy's thermodynamic identity and the Raychaudhuri equation on the null surface which was used by Jacobson has been provided for the GNC metric in [2, 1]. For the Raychaudhuri equation, we shall take a slightly different route which makes it easier to compare with Jacobson's results while the results for Paddy's thermodynamic identity will be borrowed from the above two papers.

2.1 Gaussian Null Coordinates (GNC)

This coordinate system was introduced. as far as I know, by Moncreif and Isenberg [16]. The construction of these coordinates are also discussed in [6, 26, 17, 24]. We shall briefly detail the construction and note the essential properties of this coordinate system below.

In the case of Gaussian normal coordinates near a non-null surface, the construction proceeds by using geodesics normal to the surface. This won't work for the null case, since geodesics with tangent vectors along the surface normal, say ℓ^a, actually lie on the null surface. But this offers a unique direction on the null surface and the coordinate system on the surface can be set up adapted to this direction. To do this, choose any spacelike 2-surface on the null surface and assign coordinates (x^1, x^2) on that surface. Then, carry these coordinates along the null generators of the surface with some parameter u, not necessarily affine (see Sect 4 for the case of affine parametrization), forming the third coordinate on the surface. To construct the coordinates in the region near the null surface, we introduce an auxiliary null vector k^a, satisfying $\ell_a k^a = -1$. Then, we carry the coordinates on the null surface along the null geodesics in the direction of k^a and take the fourth coordinate as an affine parameter $-r$ along these null geodesics, with $r = 0$ on the null surface. (Here, $-r$ has been used instead of r just to match with conventions we have been following.)

Then, the line element in GNC coordinates takes the following form:

$$ds^2 = -2r\alpha du^2 + 2dudr - 2r\beta_A dudx^A + q_{AB}dx^A dx^B \tag{1}$$

This line element contains six independent parameters α, β_A and q_{AB}, all dependent on all the coordinates $\left(u, r, x^A\right)$. The metric on the two-surface (i.e. $u =$ constant and $r =$ constant) is represented by q_{AB}. The surface $r = 0$ is the fiducial null surface while other $r =$constant surfaces are not null in general. There are only 6 free functions in the metric, as can be expected when we use the 4 coordinate choices to restrict 10 components of the metric for a general spacetime.

We use the symbol s_a for the normal $\partial_a r$ to the $r =$ constant surfaces. This will be a null vector on the $r = 0$ null surface. In fact, s^a will go to the null vector ℓ^a, defined earlier, on the null surface. We introduce an auxiliary null vector k^a such that $k^a s_a = -1$ everywhere. The components of these quantities in (u, r, x^A) coordinates are as follows:

$$s_a = (0,1,0,0)\,, \qquad s^a = \left(1, 2r\alpha + r^2\beta^2, r\beta^A\right) \tag{2a}$$

$$k_a = (-1,0,0,0)\,, \qquad k^a = (0,-1,0,0) \tag{2b}$$

On the null surface, we introduce two spacelike vectors $\mathbf{e_A} = (\mathbf{e_1}, \mathbf{e_2})$ which satisfy $\ell_a e^a_A = k_a e^a_A = 0$. The four vectors

$$(\ell^a, k^a, e^a_1, , e^a_2) \tag{3}$$

form a basis near the null surface. Next we introduce the vector ξ^a:

$$\xi = \frac{\partial}{\partial u} = (1,0,0,0)\ . \tag{4}$$

which goes to ℓ^a on the null surface. This vector will be called the time development vector since it corresponds to the standard time direction (which is also a Killing direction) when Schwarzschild and Rindler metrics are written in GNC form (see Appendix B in [2]). Thus, we may take it as the time corresponding to the local Rindler observers in the local Rindler frame near the null surface. We have $\xi^2 = -2r\alpha$. This is zero on the null surface, as expected since ξ^a goes to ℓ^a.

The vector $k^a = -\partial/\partial r$ is tangent to the ingoing null geodesic (ingoing since it points in the direction of decreasing r), which is affinely parametrized with affine parameter r. We denote λ_H to be the value of the affine parameter on the null surface. In the remaining discussions, we will work with λ defined through the following relation: $r = \lambda - \lambda_H$.

2.2 The Components of $F^a = G^a_b \xi^b$

In order to derive the thermodynamic identity, we focus on the vector $F^a = G^a_b \xi^b$. One way of seeing where this component comes from is to think in terms of the Noether current for gravity [2, 1] or in terms of the recently introduced gravitational momentum [22, 1]. Another way of thinking about this is to note that ξ^a goes to the null normal ℓ^a on the null surface and hence the various components of matter fluxes associated with the null surface are given by the various components of $G^a_b \xi^b$. We shall take projections of $G^a_b \xi^b$ along ℓ^a and k^a and show that these are the components used by Jacobson and Paddy respectively. The projection to the space spanned by e^a_A (see Eq. (3)) is obtained using the projector q^a_b. This gives rise to an equation of the form of the Navier-Stokes equation [1], but we shall not be discussing that result here.

2.2.1 Jacobson's Result from $G^a_b \xi^b \ell_a$ on the Null Surface

Jacobson's derivation of the Einstein equation proceeds by assuming the Clausius relation $\delta Q = T\delta S$, where the heat change δQ is taken as the matter flux across a null surface near a local equilibrium, T is the acceleration of an observer who perceives the local patch of the null surface as a local Rindler horizon and δS is the change in entropy which is proportional to the area change.

Consider the component

$$F^a \ell_a = G^a_b \xi^a \ell_b = G^{ab} \ell_a \ell_b = R^{ab} \ell_a \ell_b = 8\pi T^{ab} \ell_a \ell_b \ . \tag{5}$$

This projection actually picks up the component of F^a along k^a if you expand $F^a = A\ell^a + Bk^a + C^A e^a_A$ in the basis in Eq. (3). In GNC coordinates,

$$F^a \ell_a = R^{ab} \ell_a \ell_b = R^{rr} \ . \tag{6}$$

Evaluating this component of the Ricci tensor at the null surface $r = 0$, we obtain

$$\begin{aligned} R^{rr} &= \frac{\alpha}{\sqrt{q}} \partial_u \sqrt{q} - \partial_u^2 \ln \sqrt{q} + \frac{1}{4} \partial_u q_{AB} \partial_u q^{AB} \\ &= \alpha \partial_u \ln \sqrt{q} - \partial_u^2 \ln \sqrt{q} - \frac{1}{4} q^{AC} q^{BD} \partial_u q_{AB} \partial_u q_{CD}, \end{aligned} \tag{7}$$

where ∂_u denotes the operator $\partial/\partial u$. The induced metric on the null surface is given by $q_{ab} = g_{ab} + \ell_a k_b + k_a \ell_b$. Using q_{ab}, we can construct the second fundamental form on the null surface:

$$\Theta_{ab} = q^m_a q^n_b \nabla_m \ell_n \ . \tag{8}$$

Taking the trace of the second fundamental form, we get the expansion:

$$\Theta = g^{ab}\Theta_{ab} = q^{ab}\Theta_{ab} = q^{ab}\nabla_a \ell_b \,. \tag{9}$$

The second fundamental form and the expansion for the null surface at $r = 0$ in GNC are respectively

$$\Theta_{ab} = \frac{\partial_u q_{AB}}{2}; \quad \Theta = \frac{q^{AB}\partial_u q_{AB}}{2} = \partial_u \ln\sqrt{q} \tag{10}$$

Thus, we can write

$$F^a \ell_a = R^{rr} = \alpha\Theta - \partial_u\Theta - \Theta_{AB}\Theta^{AB} \tag{11}$$

$$= \alpha\Theta - \frac{1}{\sqrt{q}}\partial_u\left(\sqrt{q}\Theta\right) + \Theta^2 - \Theta_{AB}\Theta^{AB} \tag{12}$$

$$= \alpha\Theta - \frac{1}{\sqrt{q}}\partial_u\left(\sqrt{q}\Theta\right) - D, \tag{13}$$

where $D = \Theta_{AB}\Theta^{AB} - \Theta^2$ is identified as the dissipation corresponding to the null surface. This identification comes from the component $q^a_b F^b$ of F^a which can be written in a form similar to Navier-Stokes equations of fluid dynamics [1].

In order to compare with Jacobson's result, consider Eq. (11). We can write it in the form

$$\partial_u\Theta = \alpha\Theta - \Theta_{AB}\Theta^{AB} - R_{ab}\ell^a\ell^b \tag{14}$$

Once we decompose Θ_{ab} into its trace (expansion Θ), traceless symmetric part (shear σ_{ab}) and antisymmetric part (rotation ω_{ab}), this reduces to the form of the null Raychaudhuri equation [25] but with the first term being extra. This term appears because the null Raychaudhuri equation is usually defined with an affinely parametrized geodesic while the parameter u is not an affine parameter. If we derive the null Raychaudhuri equation without assuming affine parametrization, we can see that it is consistent with Eq. (14) [1].

But we shall follow the other route by keeping affine parametrization. Then, u has to be taken as an affine parameter in the construction of GNC. This would lead to the constraint that $\alpha = 0$ on the null surface $r = 0$ (see Sect 4). Enforcing this, Eq. (14) becomes

$$\partial_u\Theta = -\Theta_{AB}\Theta^{AB} - R_{ab}\ell^a\ell^b \tag{15}$$

This is the usual affinely parametrized null Raychaudhuri equation. In Jacobson's case, the $\Theta_{AB}\Theta^{AB}$ term is put to zero as a condition for local equilibrium. More explicitly, the rotation is zero since the null geodesics are taken to form the local horizon, while the expansion and shear are set to zero as a condition for equilibrium. If there is a given null surface with non-zero shear or expansion at a point, then these cannot be set to zero by choice of coordinates as they are geometrical quantities. But what is true is that given any point in spacetime and any null direction through that point, there is always a null surface through that point tangential to the null direction such that the expansion and shear are zero. To see this in GNC coordinates, first note that $\Theta_{ab} = \partial_u q_{ab}$. At the point P at which we want to construct the GNC coordinates on a null surface at local equilibrium, we first erect a local inertial frame (t,x,y,z). The x-axis is aligned so that the chosen null direction lies in the $x-t$ plane. If we now do a Rindler transformation, the metric can be put in the GNC form with the null surface coinciding with the local Rindler horizon with the chosen null direction along one of its generators. Since the transverse 2-metric is not touched in these transformations, they remain flat and we shall have $\partial_u q_{AB} = 0$ valid in the local inertial frame. Thus, given any point in spacetime and a null direction through it, one can always choose a null surface through the point tangential to the null direction such that $\Theta^{AB} = 0$; in other words, such that the shear, expansion and rotation are zero.

With such a choice of a null surface, Eq. (15) becomes

$$\partial_u \Theta = -R_{ab}\ell^a \ell^b \ , \tag{16}$$

which is precisely the form of the Raychaudhuri equation Jacobson integrated to get the geometric part needed to obtain the Einstein equation. Following Jacobson, we first integrate the above equation with the initial condition that $\Theta = 0$ at the point P at $u = 0$ to obtain

$$\Theta = -R_{ab}\ell^a \ell^b u \ . \tag{17}$$

Our convention is that u is increasing to the future. Jacobson's set-up involves imagining that the congruence was expanding a little to the past of P and then there was a matter flux through the congruence that provided just enough gravitational lensing to bring the expansion to zero at P. We shall integrate from the point P_0 with affine parameter value $-u_i$ to the past of P. The change in area of an infinitesimal cross-section around the chosen null generator is then given by

$$\delta A = \int_{-u_i}^{0} \Theta \sqrt{q} du d^2x = -\int_{-u_i}^{0} R_{ab}\ell^a \ell^b u \sqrt{q} du d^2x = \int_0^{u_i} R_{ab}\ell^a \ell^b \lambda \sqrt{q} d\lambda d^2x \ . \tag{18}$$

In the last step, we have changed the integration variable from u to $\lambda = -u$. Note that the change in area is positive if $R_{ab}\ell^a \ell^b > 0$, which is the condition that you obtain from the Einstein equations if the matter part satisfies the null energy condition. Finally, we assume that the entropy change that is to be associated with the null surface is proportional to its area with some proportionality constant η:

$$\delta S = \eta \delta A = \eta \int_0^{u_i} R_{ab}\ell^a \ell^b \lambda \sqrt{q} d\lambda d^2x \ . \tag{19}$$

Next, we need an accelerating observer for whom the patch of null surface acts as a local Rindler horizon. Since the proper distance of an accelerated observer from the Rindler horizon is inversely proportional to the acceleration, we shall consider a highly accelerated observer so that the observer is very close to the horizon and we can be sure that our construction of the local inertial frame and local Rindler frame are valid. In fact, we may go to the light-like limit of the infinitely accelerated observer.

In order to find the suitable observer, we shall write the GNC form with affine parametrization for the Rindler metric. In flat Milnkowski space with coordinates (t,x,y,z), the observers moving along the integral curves of the generators of boosts along the x-direction are the ones who will observe the future part of the $x=t$ null plane as a Rindler horizon. These are observers moving along the integral curves of the vector $(x,t,0,0)$. We shall try to figure out which observers in our picture can be taken to correspond to these observers. First, we shall write down the metric in the following Rindler form [27]:

$$ds^2 = -2\kappa l dT^2 + \frac{dl^2}{2\kappa l} + dy^2 + dz^2 \,. \tag{20}$$

This form is obtained from the Minkowski metric by the coordinate transformation $x = \sqrt{2l/\kappa}\cosh\kappa t$ and $t = \sqrt{2l/\kappa}\sinh\kappa t$. Defining a new coordinate U by

$$U = T + \int \frac{dl}{2\kappa l}; \quad dT = dU - \frac{dl}{2\kappa l}\,, \tag{21}$$

we transform to

$$ds^2 = -2\kappa l dU^2 + 2dUdl + dy^2 + dz^2 \,. \tag{22}$$

We have got the metric in GNC form, but U is not an affine parameter. Looking at how the vector $\partial/\partial U$ should be scaled to make it affine, we can figure out that the transformation to coordinates

$$\lambda = \frac{e^{\alpha U}}{\alpha}; \quad s = \frac{r}{\alpha\lambda} \tag{23}$$

will do. This brings the metric in the form

$$ds^2 = 2d\lambda ds + dy^2 + dz^2 \,. \tag{24}$$

I am happy to say that we have rediscovered the flat metric in the double null form in a pleasantly roundabout way. This is now of the form of affinely parametrized GNC in Eq. (39). The observer can then found to be the one moving along the integral curves of the vector $(\lambda, -s, 0)$.

Taking a cue from this, we shall look at the observers moving along the integral curves of $(u,-r,0,0)$ in the affine GNC metric of Eq. (39). So we anoint

$$\chi^a = (u, -r, 0, 0), \tag{25}$$

as the vector representing our observers. The normalized vector $\bar{\chi}^a$ will be the four-velocities. Note that, on the null surface $r=0$, we have

$$\chi^a = u\frac{\partial}{\partial u} = u\ell^a \ . \tag{26}$$

Writing $\bar{\chi}^a = N\chi^a$, the acceleration of the trajectories will be given by N and the corresponding acceleration temperature will be

$$T = \frac{N}{2\pi} \tag{27}$$

near the null surface.

The momentum four-vector associated to the matter flux by one such observer will be $NT^{ab}\chi_b$. The energy of the matter flux across the null surface as observed by this observer is the heat change

$$\delta Q = \int_{-u_i}^{0} NT^{ab}\chi_b\ell_a\sqrt{q}dud^2x = N\int_{-u_i}^{0} uT^{ab}\ell_b\ell_a\sqrt{q}dud^2x, \tag{28}$$

where we have used Eq. (26). Changing variables to $\lambda = -u$ as in Eq. (19), we obtain

$$\delta Q = N\int_{0}^{u_i} \lambda T^{ab}\ell_b\ell_a\sqrt{q}d\lambda d^2x \ . \tag{29}$$

Now we have all the ingredients in place. Demanding $\delta Q = T\delta S$ and using Eq. (19), Eq. (27) and Eq. (29), we obtain the condition

$$R_{ab}\ell^a\ell^b = -\frac{2\pi}{\eta}T_{ab}\ell^a\ell^b \ . \tag{30}$$

This is almost right, except for a pesky minus sign. If we obtain Einstein equation from the above relation, η will be set as negative and this would mean that area increase is entropy decrease from Eq. (19). We have been careful with signs around Eq. (19), so let us look back at Eq. (29). If we enforce the null energy condition $T^{ab}\ell_a\ell_b > 0$, the heat turns out to be negative. Since we require a positive heat change to correspond to the positive change in entropy, we redefine

$$\delta Q = -N\int_{0}^{u_i} \lambda T^{ab}\ell_b\ell_a\sqrt{q}d\lambda d^2x \ . \tag{31}$$

This is in fact the correct definition. The original source of the extra minus sign was the fact that $\chi^a = u\partial/\partial u$ is past-directed in our region of integration since u is negative, and hence $-\chi^a$ should have been used.

Thus, we obtain the equation

$$R_{ab}\ell^a\ell^b = \frac{2\pi}{\eta}T_{ab}\ell^a\ell^b \ . \tag{32}$$

From here, we can follow Jacobson [8] to obtain the full Einstein equations.

2.2.2 Paddy's Result from $G^a_b \xi^b k_a$ on the Null Surface

Next, we shall discuss Paddy's approach where a certain component of the Einstein equations near a horizon takes a form similar to the first law of thermodynamics [18, 12, 11, 2, 1, 10]. In Paddy's approach the Einstein equations are not derived, but it is shown that a certain component of Einstein equations projected on a null surface has a thermodynamic interpretation. Note that this difference is superficial since even in Paddy's case one can derive the Einstein equations by starting from the thermodynamic identity backward and demanding that it holds for all null surfaces and even in Jacobson's case one can prove the Clausius relation starting from the Einstein equations.

To obtain Paddy's result, we look at another component of F^a. We shall take its projection along the auxiliary null vector k_a. Then, we obtain

$$F^a k_a = G^a_b \xi^b k_a = 8\pi T^a_b \xi^b k_a \,. \tag{33}$$

In this case, the work has already been done in [2, 1] and hence we just borrow the results. Working in GNC, the component $G^a_b \xi^b k_a = 8\pi T^a_b \xi^b k_a$ is interpreted and written in the form

$$\bar{F}\delta\bar{\lambda} = T\delta_{\bar{\lambda}} S - \delta_{\bar{\lambda}} E \,. \tag{34}$$

Here, F is the integral of $T^a_b \xi^b k_a$ over the null surface, interpreted as the force acting on the patch of the null surface and $\delta\lambda = \delta r$ is a small shift of the horizon in the direction of k^a. This is a small shift in the r-direction in GNC coordinates. Note that r is also an affine parameter.

On the RHS, $T = \alpha/2\pi$ is the acceleration temperature corresponding to the observers moving along the integral curves of ξ^a near $r = 0$ with the assumption that α is slowly varying in time. More precisely, we assume $\partial_u \alpha \ll \alpha^2$. The change in entropy δS is just the change in the 2-surface area, with appropriate factors, when the surface is shifted outward. The change in area is the integral of $\partial_r \sqrt{q}\delta r$ integrated over the patch of the null surface.

Finally, we have the change in energy. The quantity E here is given by the expression

$$E = \frac{1}{16\pi}\int dr \int d^2x\, R^{(2)} - \frac{1}{8\pi}\int d^2x \partial_u \sqrt{q} - \frac{1}{16\pi}\int dr \int d^2x \sqrt{q}\left\{\frac{1}{2}\beta_A \beta^A\right\} \,. \tag{35}$$

Here, one term has been put to zero under the assumption that the u-constant, r-constant 2-surface on the null surface is closed. The identification of this quantity as the energy is due to the fact that it is able to reproduce the known expressions of energy in several known cases. For example, this reduces to the mass for the Schwarzschild metric.

With these identifications, Eq. (34) is of the form of the first law of thermodynamics.

3 Discussion

At first sight, both Paddy's results and Jacobson's results look very similar. Jacobson's result says that one can derive Einstein equations from $\delta Q = T\delta S$ imposed on local Rindler horizons, which can be constructed along any null direction around any point in spacetime. Paddy's result is that a certain projection of the Einstein equations on a null surface can be interpreted to be of the form $\delta E = T\delta S - F\delta\lambda$. In thermodynamics, these relation will be called the Clausius relation and the first law of thermodynamics. In fact, one may be tempted to put both together and obtain the equation $\delta E = \delta Q - F\delta\lambda$. But closer scrutiny reveals that things are not that simple.

The apparent point of conflict is that both δQ in Jacobson's case and δE in Paddy's case are components of matter energy-momentum flux across the horizon. Since identification of which component is which physically is a little complicated near the null surface, I am sure many people must have thought that these are the same components. Once this is assumed, there appears to be a discrepancy with Paddy having an extra term compared to Jacobson's starting point, since the $T\delta S$ terms seem unambiguous.

I hope I have shed some light on this issue in this article, building up on work previously done in [2, 1]. Working in the framework of Gaussian null coordinates, one can see that

1. Paddy's result and Jacobson's result come from two different components of the Einstein tensor, and equivalently of the matter energy-momentum tensor, near the null surface.
2. The entropy change in the $T\delta S$ term is not the same in the two cases. In Paddy's case, the change is from the change in area along the null geodesics off the null surface, while the change in Jacobson's case is along the null geodesics on the null surface.

Another paper which compared the two approaches is [9]. It is not clear how the results there compare to the results stated here. The discussion in [9] had the general static metric introduced in [13, 14] as the reference metric although the final results are stated in tensorial form. It will be interesting to see how these results look when translated to Gaussian null coordinates.

Epilogue

> Theoretical physics is fun. Most of us indulge in it for the same reason a painter paints or a dancer dances...Occasionally, there are additional benefits like fame and glory and even practical uses; but most good theoretical physicists will agree that these are not the primary

reasons why they are doing it. The fun in figuring out the solutions to Nature's brain teasers is a reward in itself.

Source- Paddy's book [20]
My first encounter- After PhD

Acknowlegements

I am happy to dedicate this article to Paddy, who I was fortunate to have as my PhD supervisor, on the occasion of his 60th birthday as a tribute to his never flagging enthusiasm for anything under the sun, his ocean-deep knowledge and his unmatched tenacity. I also acknowledge discussions with Sumanta Chakraborty, Bibhas Majhi and Dawood Kothawala on related topics in the past.

Appendices

4 Gaussian Null Coordinates with Affine Parametrization

The line element in GNC coordinates was given in Eq. (1) as

$$ds^2 = -2r\alpha du^2 + 2dudr - 2r\beta_A dudx^A + q_{AB}dx^A dx^B \ , \tag{36}$$

Here, α, β_A and q_{AB} are arbitrary functions. This was derived by taking u to be an arbitrary parameter along the null geodesics on the null surface $r = 0$ [24]. But suppose we now impose the condition that u is an affine parameter on the null surface. This is equivalent to the condition that $\xi^a \nabla_a \xi^b = 0$ at $r = 0$ for $\xi^a = \partial/\partial u$. For the above line element, we have

$$\xi^a \nabla_a \xi^b = \Gamma^b_{ac}\xi^a\xi^c = \Gamma^b_{uu} \ . \tag{37}$$

Equating this to zero, we get the following conditions at $r = 0$:

$$\partial_u g_{uu} = 0; \ \partial_r g_{uu} = 0; \ \partial_A g_{uu} = 0 \ . \tag{38}$$

Since $g_{uu} = -2r\alpha$, the first and third conditions are automatically satisfied, while the second condition implies $\alpha = 0$ at $r = 0$. This can be enforced by putting $\alpha = r\gamma$ where γ is an arbitrary function. Thus, the form of the GNC line element with affine parametrization is

$$ds^2 = -2r^2\gamma du^2 + 2dudr - 2r\beta_A dudx^A + q_{AB}dx^A dx^B \ . \tag{39}$$

Note that this form was indicated in [17] where the R_{uu} component was compared to the Raychaudhuri equation to claim that α has to be proportional to r. It is not clear from the text whether the author realized that this is necessary only if u is taken to be affine. But since affine parametrization can always be taken, it is true that the GNC metric can always be written in the above form.

References

1. Chakraborty, S., Padmanabhan, T.: Thermodynamical interpretation of the geometrical variables associated with null surfaces. Phys. Rev. **D92**(10), 104,011 (2015). DOI 10.1103/PhysRevD.92.104011
2. Chakraborty, S., Parattu, K., Padmanabhan, T.: Gravitational field equations near an arbitrary null surface expressed as a thermodynamic identity. JHEP **10**, 097 (2015). DOI 10.1007/JHEP10(2015)097
3. Chirco, G., Liberati, S.: Non-equilibrium Thermodynamics of Spacetime: The Role of Gravitational Dissipation. Phys. Rev. **D81**, 024,016 (2010). DOI 10.1103/PhysRevD.81.024016
4. Dey, R., Liberati, S., Mohd, A.: Higher derivative gravity: field equation as the equation of state (2016)
5. Eling, C., Guedens, R., Jacobson, T.: Non-equilibrium thermodynamics of spacetime. Phys. Rev. Lett. **96**, 121,301 (2006). DOI 10.1103/PhysRevLett.96.121301
6. Friedrich, H., Racz, I., Wald, R.M.: On the rigidity theorem for space-times with a stationary event horizon or a compact Cauchy horizon. Commun.Math.Phys. **204**, 691–707 (1999). DOI 10.1007/s002200050662
7. Guedens, R., Jacobson, T., Sarkar, S.: Horizon entropy and higher curvature equations of state. Phys. Rev. **D85**, 064,017 (2012). DOI 10.1103/PhysRevD.85.064017
8. Jacobson, T.: Thermodynamics of space-time: The Einstein equation of state. Phys.Rev.Lett. **75**, 1260–1263 (1995). DOI 10.1103/PhysRevLett.75.1260
9. Kothawala, D.: The thermodynamic structure of Einstein tensor. Phys.Rev. **D83**, 024,026 (2011). DOI 10.1103/PhysRevD.83.024026
10. Kothawala, D.: Accelerated Observers, Thermal Entropy, and Spacetime Curvature (2016)
11. Kothawala, D., Padmanabhan, T.: Thermodynamic structure of Lanczos-Lovelock field equations from near-horizon symmetries. Phys. Rev. **D79**, 104,020 (2009). DOI 10.1103/PhysRevD.79.104020
12. Kothawala, D., Sarkar, S., Padmanabhan, T.: Einstein's equations as a thermodynamic identity: The Cases of stationary axisymmetric horizons and evolving spherically symmetric horizons. Phys.Lett. **B652**, 338–342 (2007). DOI 10.1016/j.physletb.2007.07.021
13. Medved, A., Martin, D., Visser, M.: Dirty black holes: Space-time geometry and near horizon symmetries. Class.Quant.Grav. **21**, 3111–3126 (2004). DOI 10.1088/0264-9381/21/13/003
14. Medved, A., Martin, D., Visser, M.: Dirty black holes: Symmetries at stationary nonstatic horizons. Phys.Rev. **D70**, 024,009 (2004). DOI 10.1103/PhysRevD.70.024009
15. Mohd, A., Sarkar, S.: Thermodynamics of Local Causal Horizons. Phys. Rev. **D88**(2), 024,026 (2013). DOI 10.1103/PhysRevD.88.024026
16. Moncrief, V., Isenberg, J.: Symmetries of cosmological cauchy horizons. Communications in Mathematical Physics **89**(3), 387–413 (1983). DOI 10.1007/BF01214662. URL http://dx.doi.org/10.1007/BF01214662
17. Morales, E.M.: On a Second Law of Black Hole Mechanics in a Higher Derivative Theory of Gravity. http://www.theorie.physik.uni-goettingen.de/forschung/qft/theses/dipl/Morfa-Morales.pdf (2008)

18. Padmanabhan, T.: Classical and quantum thermodynamics of horizons in spherically symmetric space-times. Class.Quant.Grav. **19**, 5387–5408 (2002). DOI 10.1088/0264-9381/19/21/306
19. Padmanabhan, T.: Thermodynamical Aspects of Gravity: New insights. Rept. Prog. Phys. **73**, 046,901 (2010). DOI 10.1088/0034-4885/73/4/046901
20. Padmanabhan, T.: Sleeping Beauties in Theoretical Physics, vol. 895. Springer (2015)
21. Padmanabhan, T.: Exploring the Nature of Gravity. In: The Planck Scale II Wroclaw, Poland, September 7-12, 2015 (2016). URL https://inspirehep.net/record/1419624/files/arXiv:1602.01474.pdf
22. Padmanabhan, T.: Momentum density of spacetime and the gravitational dynamics. Gen. Rel. Grav. **48**(1), 4 (2016). DOI 10.1007/s10714-015-1996-z
23. Padmanabhan, T.: The Atoms Of Space, Gravity and the Cosmological Constant. Int. J. Mod. Phys. **D25**(07), 1630,020 (2016). DOI 10.1142/S0218271816300202
24. Parattu, K., Chakraborty, S., Majhi, B.R., Padmanabhan, T.: A Boundary Term for the Gravitational Action with Null Boundaries. Gen. Rel. Grav. **48**(7), 94 (2016). DOI 10.1007/s10714-016-2093-7
25. Poisson, E.: A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics, 1st edn. Cambridge University Press (2007)
26. Racz, I.: Stationary Black Holes as Holographs. Class.Quant.Grav. **24**, 5541–5572 (2007). DOI 10.1088/0264-9381/24/22/016
27. T.Padmanabhan: Gravitation: Foundations and Frontiers. Cambridge University Press, Cambridge, UK (2010)
28. Wald, R.M.: General Relativity, 1st edn. The University of Chicago Press (1984)